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A new method for helicity calculations [1]

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Abstract

We propose a new helicity formalism based on the formal insertion in spinor lines of a complete set of states build up with unphysical spinors. The method is developed both for massless and massive fermions for which it turns out to be particularly fast. All relevant formulae are given.

Introduction

In high energy collisions many particles or partons widely separated in phase space are often produced. The calculation of cross sections for these processes is made difficult by the large number of Feynman diagrams which appear in the perturbative expansion. This is due both to the complexity of non abelian theories and to simple combinatorics, which generates more and more diagrams when the number of external particles grows.

If one computes unpolarized cross sections for a process described by many Feynman diagrams with the textbook method of considering the amplitude modulus squared and taking the traces, one can end up with a prohibitive number of traces to evaluate. The calculation becomes simpler if one uses the so called helicity-amplitudes techniques. In such an approach, given an assigned helicity to external particles, one computes the contribution of every single diagram k as a complex number a_k, sums over all k's and takes the modulus squared. To obtain unpolarized cross sections one simply sums the modulus squared for the various external helicities.

The use of helicity amplitude techniques in high energy physics dates back to ref. [1, 2]. Many different approaches have been developed [3-12], and even a brief survey of the vast literature on this subject goes far beyond the scope of this paper.

Two among the most popular schemes are those of ref. [7, 10]. They can be used both with massless and massive particles. We have employed them in the past for several calculations. One of them is based [10] on the choice of a specific representation for the Dirac matrices. It is then possible to obtain explicit expressions for the spinors, for the matrices at the vertices and for the fermion propagators. The complex number corresponding to a given diagram is obtained multiplying these matrices. In the other one [7, 12] all spinors for any physical momentum are defined in term of a basic spinor for an auxiliary lightlike momentum. Decomposing the internal momenta in terms of the external ones, and using the fact that $\sum_\lambda u\bar{u} = \not{p} + m$, $\sum_\lambda v\bar{v} = \not{p} - m$, all spinor lines are reduced to an algebraic combination of spinors products $\bar{u}(p_1)u(p_2)$. In order to use this method the polarization vectors of spin-1 particles have to be expressed as $\bar{u}\gamma^\mu u$ currents.

In the rest of the paper we describe an approach to helicity amplitudes which is based on the formal insertion in spinor lines of a complete set of states build up with unphysical spinors. This new formalism, that we have tested in several physical computations, combines the best features of [7, 10], is highly flexible, has a modular structure and in our experience is faster than previous methods.

Spinors

The method we will describe in the following sections makes use of generalized spinors $u(p)$ which coincide with the usual ones when $p^2 \geq 0$, but are defined also for spacelike momenta. We discuss such a generalization in this section.

Spinors may be defined as eigenstates of $\not{p}$. This is equivalent to say that they must satisfy Dirac equations:

$$\not{p}u(p) = +mu(p) \qquad\qquad \not{p}v(p) = -mv(p) \tag{1}$$

Since $\not{p}$ is not hermitian, m need not to be real, but it is constrained to satisfy $m^2 = p^2$ as $\not{p}^2 = p^2$. When $p^2 \leq 0$, the two eigenvalues are imaginary and one can choose, for instance, to associate $u(p)$ with the eigenvalue m whose imaginary part is positive. The remaining degeneracy is normally eliminated considering eigenstates of $\gamma^5 \not{s}$:

$$\gamma^5 \not{s} u(p, +) = u(p, +) \qquad\qquad \gamma^5 \not{s} u(p, -) = -u(p, -)$$
$$\gamma^5 \not{s} v(p, +) = v(p, +) \qquad\qquad \gamma^5 \not{s} v(p, -) = -v(p, -) \tag{2}$$

where the polarization vector s has to satisfy the two conditions

$$s \cdot p = 0 \qquad\qquad s^2 = -1 \tag{3}$$

The first one implies that $\gamma^5 \not{s}$ commutes with $\not{p}$, the second that $(\gamma^5 \not{s})^2 = 1$ and hence that its two eigenvalues are ± 1.

One can easily costruct an example of spinors satisfying eqs.(1,2) for any value of p^2 with a straightforward generalization of the method of ref.[7]. In such a method, one first defines spinors $w(k_0, \lambda)$ for an auxiliary massless vector k_0 satisfying

$$w(k_0, \lambda) \bar{w}(k_0, \lambda) = \frac{1 + \lambda \gamma_5}{2} \not{k}_0 \tag{4}$$

and with their relative phase fixed by

$$w(k_0, \lambda) = \lambda \not{k}_1 w(k_0, -\lambda), \tag{5}$$

with k_1 a second auxiliary vector such that $k_1^2 = -1$, $k_0 \cdot k_1 = 0$. Spinors for a four momentum p, with $m^2 = p^2$ are then obtained as:

$$u(p, \lambda) = \frac{\not{p} + m}{\sqrt{2\, p \cdot k_0}}\, w(k_0, -\lambda) \qquad v(p, \lambda) = \frac{\not{p} - m}{\sqrt{2\, p \cdot k_0}}\, w(k_0, -\lambda) \tag{6}$$

One can readily check that u and v of eq.(6) satisfy eqs.(1, 2) also when $p^2 \leq 0$ and m is imaginary.

Some care must be taken in defining in the general case the conjugate spinors $\bar{u}(p, \lambda)$, $\bar{v}(p, \lambda)$. In fact the usual quantities $\bar{u} = u^\dagger \gamma^0$, $\bar{v} = v^\dagger \gamma^0$ do not satisfy normal Dirac equations

$$\bar{u}(p) \not{p} = +m\bar{u}(p) \qquad\qquad \bar{v}(p) \not{p} = -m\bar{v}(p) \tag{7}$$

when m is imaginary. Taking the hermitiean conjugate of eqs.(1), one readily verifies on the contrary that $\bar{u} = v^\dagger \gamma_0$, $\bar{v} = u^\dagger \gamma_0$ do.

We define $\bar{u}(p, \lambda)$, $\bar{v}(p, \lambda)$ as the spinors satisfiyng both eqs.(7) and

$$\bar{u}(p, +)\gamma^5 \not{s} = \bar{u}(p, +) \qquad\qquad \bar{u}(p, -)\gamma^5 \not{s} = -\bar{u}(p, -)$$
$$\bar{v}(p, +)\gamma^5 \not{s} = \bar{v}(p, +) \qquad\qquad \bar{v}(p, -)\gamma^5 \not{s} = -\bar{v}(p, -) \tag{8}$$

With this definition the usual orthogonality relations among spinors are satisfied. As a consequence, choosing the normalization

$$\bar{u}(p, \lambda)u(p, \lambda) = 2m \qquad\qquad \bar{v}(p, \lambda)v(p, \lambda) = -2m, \tag{9}$$

the completeness relation

$$1 = \sum_\lambda \frac{u(p,\lambda)\bar{u}(p,\lambda) - v(p,\lambda)\bar{v}(p,\lambda)}{2m} \qquad (10)$$

is verified for any value of p^2.

The explicit relation between u, v and $\bar{u}$, $\bar{v}$ is

$$\bar{u}(p,\lambda) = u^\dagger(p,\lambda)\gamma^0 \qquad \bar{v}(p,\lambda) = v^\dagger(p,\lambda)\gamma^0 \qquad (11)$$

as usual when $p^2 > 0$. For $p^2 < 0$ one has

$$\bar{u}(p,\lambda) = v^\dagger(p,\lambda)\gamma^0 \qquad \bar{v}(p,\lambda) = u^\dagger(p,\lambda)\gamma^0 \qquad (12)$$

if the components of the polarization vector s are real, and

$$\bar{u}(p,\lambda) = v^\dagger(p,-\lambda)\gamma^0 \qquad \bar{v}(p,\lambda) = u^\dagger(p,-\lambda)\gamma^0 \qquad (13)$$

if they become purely imaginary for imaginary m. This is precisely the case when one uses the spinors of ref.[7]: the polarization vector is $s^\mu = p^\mu/m - (m/p{\cdot}k_0)k_0^\mu$ and for a spacelike momentum p it is equal to the imaginary unit times a timelike vector. It is interesting to notice that for $k_0^0 = \alpha(p^0 - |p|)/p^2$ and $\vec{k}_0 = \alpha\vec{p}(|p| - p^0)/|p|p^2$, with α an arbitrary factor, one recovers the usual helicity polarization vector $s = (|p|^2, p_0\vec{p})/m|p|$.

With the previous definitions, the spinors conjugate to u and v defined in eq.(6) are given by the simple formulae

$$\bar{u}(p,\lambda) = \bar{w}(k_0,-\lambda)\frac{\slashed{p} + m}{\sqrt{2\,p{\cdot}k_0}} \qquad \bar{v}(p,\lambda) = \bar{w}(k_0,-\lambda)\frac{\slashed{p} - m}{\sqrt{2\,p{\cdot}k_0}} \qquad (14)$$

in all cases. Moreover, making use of spinors of such a kind to compute amplitudes, one has never in practice to worry about which determination of $\sqrt{p^2}$ and $\sqrt{2\,p{\cdot}k_0}$ to use when p is spacelike. As we will see in the next sections in fact, only the square of the above quantities appear at the end of the computation.

Outline of the method. T functions

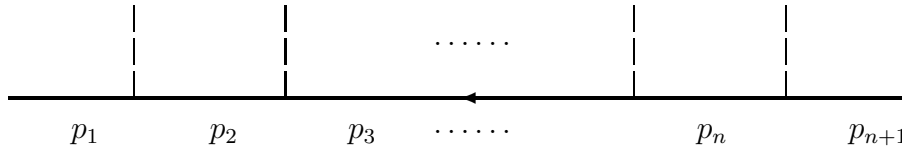The spinor part of every massive fermion line with n insertions

Fig. 1

has a generic expression of the following type

$$T^{(n)} = \overline{U}(p_1,\lambda_1)\chi_1(\slashed{p}_2 + \mu_2)\chi_2(\slashed{p}_3 + \mu_3)\cdots(\slashed{p}_n + \mu_n)\chi_n U(p_{n+1},\lambda_{n+1}) \qquad (15)$$

where λ_1 and λ_{n+1} are the polarizations of the external fermions, p_1 and p_{n+1} their momenta. $p_2, \ldots, p_n$ and $\mu_2, \ldots, \mu_n$ are the 4-momenta and masses appearing in the fermion propagators. $U(p, \lambda)$ ($\overline{U}(p, \lambda)$) stands for either $u(p, \lambda)$ ($\bar{u}(p, \lambda)$) or $v(p, \lambda)$ ($\bar{v}(p, \lambda)$). The χ's are

$$\chi_i^S \equiv \chi^S(c_{r_i}, c_{l_i}) = c_{r_i} \left(\frac{1 + \gamma_5}{2} \right) + c_{l_i} \left(\frac{1 - \gamma_5}{2} \right) \tag{16}$$

when the insertion corresponds to a scalar (or pseudoscalar), or

$$\chi_i^V \equiv \chi^V(\eta_i, c_{r_i}, c_{l_i}) = \slashed{\eta}_i \left[c_{r_i} \left(\frac{1 + \gamma_5}{2} \right) + c_{l_i} \left(\frac{1 - \gamma_5}{2} \right) \right] \tag{17}$$

when it corresponds to a vector particle whose 'polarization' is η. Of course η can be the polarization vector of the external particle or the vector resulting from a complete subdiagram which is connected in the i–th position to the fermion line.

Let us start considering the case in which there are only two insertions:

$$T^{(2)}(p_1; \eta_1, c_1; p_2; \eta_2, c_2; p_3) = \overline{U}(p_1, \lambda_1) \chi_1 (\slashed{p}_2 + \mu_2) \chi_2 U(p_3, \lambda_3). \tag{18}$$

Here and in the following we indicate with c_i both the couplings c_{r_i} and c_{l_i}. Obviously the vectors η only appear as an argument for vector insertions.

Even if $m_2^2 \equiv p_2^2$ does not correspond to the mass of any physical particle, one can insert in eq. (18), just before $(\slashed{p}_2 + \mu_2)$ a complete set of states in the form:

$$1 = \sum_\lambda \frac{u(p_2, \lambda)\bar{u}(p_2, \lambda) - v(p_2, \lambda)\bar{v}(p_2, \lambda)}{2m_2} \tag{19}$$

and make use of Dirac equations to get:

$$\begin{aligned}
T^{(2)} &= \frac{1}{2} \sum_{\lambda_2} \left(\overline{U}(p_1, \lambda_1) \chi_1 u(p_2, \lambda_2) \times \bar{u}(p_2, \lambda_2) \chi_2 U(p_3, \lambda_3) \times \left(1 + \frac{\mu_2}{m_2} \right) \right. \\
&\left. + \overline{U}(p_1, \lambda_1) \chi_1 v(p_2, \lambda_2) \times \bar{v}(p_2, \lambda_2) \chi_2 U(p_3, \lambda_3) \times \left(1 - \frac{\mu_2}{m_2} \right) \right)
\end{aligned} \tag{20}$$

This example can be generalized to any number of insertions and shows that the factors $(\slashed{p}_i + \mu_i)$ can be easily eliminated, reducing all fermion lines essentially to products of T functions:

$$T_{\lambda_1 \lambda_2}(p_1; \eta, c; p_2) = \overline{U}(p_1, \lambda_1) \chi U(p_2, \lambda_2) \tag{21}$$

defined for any value of p_1^2 and p_2^2.

It is convenient to use the spinors $u(p, \lambda)$ and $v(p, \lambda)$ defined in eqs.(6,14). With this choice, to which we will adhere from now on, the T functions (21) have a simple dependence on m_1 and m_2 and as a consequence the rules for constructing spinor lines out of them are simple. Every T function has in fact an expression of the following kind:

$$\widetilde{T}_{\lambda_1 \lambda_2}(p_1; \eta, c; p_2) \equiv \sqrt{p_1 \cdot k_0} \sqrt{p_2 \cdot k_0} \, T_{\lambda_1 \lambda_2}(p_1; \eta, c; p_2) = \tag{22}$$

$$A_{\lambda_1\lambda_2}(p_1;\eta,c;p_2) + M_1 B_{\lambda_1\lambda_2}(p_1;\eta,c;p_2) + M_2 C_{\lambda_1\lambda_2}(p_1;\eta,c;p_2) + M_1 M_2 D_{\lambda_1\lambda_2}(p_1;\eta,c;p_2)$$

where

$$
\begin{aligned}
M_i &= +m_i && \text{if} && U(p_i,\lambda_i) = u(p_i,\lambda_i) \\
M_i &= -m_i && \text{if} && U(p_i,\lambda_i) = v(p_i,\lambda_i).
\end{aligned}
\tag{23}
$$

The functions A, B, C, D turn out to be independent of m_1 and m_2 and of the u or v nature of $\overline{U}(p_1,\lambda_1)$ and $U(p_2,\lambda_2)$. We give in Appendix A the expressions for A^V, B^V, C^V, D^V and A^S, B^S, C^S, D^S, which are the A, B, C, D functions for a vector and a scalar insertion respectively .

Spinor lines

We will show in this section how to compute recursively the functions

$$\widetilde{T}^{(n)} = T^{(n)}\sqrt{p_1 \cdot k_0}\ \sqrt{p_{n+1} \cdot k_0}\ (p_2 \cdot k_0)\ (p_3 \cdot k_0) \cdots (p_n \cdot k_0). \tag{24}$$

from which the $T^{(n)}$ themselves can then be immediately obtained at the end of the computation, dividing by the appropriate factors.

Let us denote with $\widetilde{T}, A, B, C, D$ the 2x2 matrices whose elements are $\widetilde{T}_{\lambda_1\lambda_2}$, $A_{\lambda_1\lambda_2}$, $B_{\lambda_1\lambda_2}$, $C_{\lambda_1\lambda_2}$, $D_{\lambda_1\lambda_2}$. With this notation, making use of eqs. (22) and (21), eq. (20) reads:

$$
\begin{aligned}
\widetilde{T}^{(2)}(1,2,3) = \frac{1}{2}\Big[&\Big(A(1,2) + M_1 B(1,2) + m_2 C(1,2) + M_1 m_2 D(1,2)\Big) \\
&\times \Big(1 + \frac{\mu_2}{m_2}\Big) \times \Big(A(2,3) + m_2 B(2,3) + M_3 C(2,3) + m_2 M_3 D(2,3)\Big) \\
&+ \Big(A(1,2) + M_1 B(1,2) - m_2 C(1,2) - M_1 m_2 D(1,2)\Big) \\
&\times \Big(1 - \frac{\mu_2}{m_2}\Big) \times \Big(A(2,3) - m_2 B(2,3) + M_3 C(2,3) - m_2 M_3 D(2,3)\Big)\Big]
\end{aligned}
\tag{25}
$$

where we have used the shorthands $(1,2)$ and $(1,2,3)$ for $(p_1;\eta_1,c_1;p_2)$ and $(p_1;\eta_1,c_1;p_2;\eta_2,c_2;p_3)$ respectively.

Elementary algebra shows that $\widetilde{T}^{(2)}$ has again the same dependence on the external (possibly unphysical) masses as in (22):

$$\widetilde{T}^{(2)}(1,2,3) = A^{(2)}(1,2,3) + M_1 B^{(2)}(1,2,3) + M_3 C^{(2)}(1,2,3) + M_1 M_3 D^{(2)}(1,2,3) \tag{26}$$

with

$$
\begin{aligned}
A^{(2)}(1,2,3) &= A(1,2)\Big(A(2,3) + \mu_2 B(2,3)\Big) + C(1,2)\Big(\mu_2 A(2,3) + p_2^2 B(2,3)\Big) \\
B^{(2)}(1,2,3) &= B(1,2)\Big(A(2,3) + \mu_2 B(2,3)\Big) + D(1,2)\Big(\mu_2 A(2,3) + p_2^2 B(2,3)\Big) \\
C^{(2)}(1,2,3) &= A(1,2)\Big(C(2,3) + \mu_2 D(2,3)\Big) + C(1,2)\Big(\mu_2 C(2,3) + p_2^2 D(2,3)\Big) \\
D^{(2)}(1,2,3) &= B(1,2)\Big(C(2,3) + \mu_2 D(2,3)\Big) + D(1,2)\Big(\mu_2 C(2,3) + p_2^2 D(2,3)\Big)
\end{aligned}
\tag{27}
$$

This implies that $A^{(2)}$, $B^{(2)}$, $C^{(2)}$, $D^{(2)}$ can be reinserted in an equation like eq. (25) to give the $\widetilde{T}$ function $\widetilde{T}^{(3)}$ corresponding to a fermion line with 3 insertions, and so on. So one can generalize eqs. (25,26,27) by induction: every $\widetilde{T}^{(i)}$ turns out to be of the form

$$\widetilde{T}^{(i)} = A^{(i)} + M_1 B^{(i)} + M_{i+1} C^{(i)} + M_1 M_{i+1} D^{(i)} \tag{28}$$

and from the knowledge of $A^{(i)}(1,\ldots,i)$, $B^{(i)}(1,\ldots,i)$, $C^{(i)}(1,\ldots,i)$, $D^{(i)}(1,\ldots,i)$ and $A^{(j)}(i+1,\ldots,i+j)$, $B^{(j)}(i+1,\ldots,i+j)$, $C^{(j)}(i+1,\ldots,i+j)$, $D^{(j)}(i+1,\ldots,i+j)$, one gets:

$$\begin{aligned}
A^{(i+j)} &= A^{(i)}\left(A^{(j)} + \mu_i B^{(j)}\right) + C^{(i)}\left(\mu_i A^{(j)} + p_i^2 B^{(j)}\right) \\
B^{(i+j)} &= B^{(i)}\left(A^{(j)} + \mu_i B^{(j)}\right) + D^{(i)}\left(\mu_i A^{(j)} + p_i^2 B^{(j)}\right) \\
C^{(i+j)} &= A^{(i)}\left(C^{(j)} + \mu_i D^{(j)}\right) + C^{(i)}\left(\mu_i C^{(j)} + p_i^2 D^{(j)}\right) \\
D^{(i+j)} &= B^{(i)}\left(C^{(j)} + \mu_i D^{(j)}\right) + D^{(i)}\left(\mu_i C^{(j)} + p_i^2 D^{(j)}\right)
\end{aligned} \tag{29}$$

which build up $\widetilde{T}^{(i+j)}(1,\ldots,i+j)$. In eq. (29) μ_i and p_i are the mass and momentum of the propagator which connects the left i insertions with the right j ones

It should by now be evident that the evaluation of any spinor line can be performed computing the A, B, C, D matrices relative to every single insertion and combining them toghether with the help of eq. (29) until one gets to the final $T^{(n)}$. It is important to point out that the unphysical masses m_i $(i = 2, \cdots, n)$ do not appear in eqs. (27,28,29): only their squares p_i^2 do. Therefore,as anticipated, the determination of $m_i = \sqrt{p_i^2}$ is irrelevant. The same conclusion can be drawn for $\sqrt{p_i \cdot k_0}$ from eq. (24) and the fact that the expressions (A.2) are independent of square roots.

τ matrices

It is convenient to cast the previous formulae in a matrix notation. We drop the superscripts (n) when not necessary.

Every piece of a spinor line as well as every complete spinor line with n insertions is completely known when we know the matrix

$$\tau = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \tag{30}$$

The law of composition of two pieces of spinor line , connected by a fermion propagator with 4-momentum p and mass μ, whose matrices are

$$\tau_1 = \begin{pmatrix} A_1 & C_1 \\ B_1 & D_1 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} A_2 & C_2 \\ B_2 & D_2 \end{pmatrix},$$

is simply (cfr. eq. (29)) :

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} A_1 & C_1 \\ B_1 & D_1 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ \mu & p^2 \end{pmatrix} \begin{pmatrix} A_2 & C_2 \\ B_2 & D_2 \end{pmatrix} \tag{31}$$

We will sometimes indicate the above composition as follows:

$$\tau = \tau_1 \bullet \tau_2 \tag{32}$$

If we call π_i the matrix

$$\pi_i = \begin{pmatrix} 1 & \mu_i \\ \mu_i & p_i^2 \end{pmatrix} \tag{33}$$

corresponding to the propagator of 4-momentum p_i and τ_i the matrix associated with the i-th insertion of fig. 1, the τ matrix of the whole spinor line can then be computed as follows:

$$\tau = \tau_1 \pi_2 \tau_2 \pi_3 \tau_3 \cdots \pi_{n-1} \tau_{n-1} \pi_n \tau_n \tag{34}$$

Explicitely written as 4×4 matrices, τ and π_i are:

$$\tau = \begin{pmatrix} A_{++} & A_{+-} & C_{++} & C_{+-} \\ A_{-+} & A_{--} & C_{-+} & C_{--} \\ B_{++} & B_{+-} & D_{++} & D_{+-} \\ B_{-+} & B_{--} & D_{-+} & D_{--} \end{pmatrix} \qquad \pi_i = \begin{pmatrix} 1 & 0 & \mu_i & 0 \\ 0 & 1 & 0 & \mu_i \\ \mu_i & 0 & p_i^2 & 0 \\ 0 & \mu_i & 0 & p_i^2 \end{pmatrix} \tag{35}$$

From the espressions of A, B, C, D given in appendix A, one can see that the τ's of a single vector or scalar insertion have the particular form:

$$\tau^V = \begin{pmatrix} A_{++}^V & 0 & 0 & C_{+-}^V \\ 0 & A_{--}^V & C_{-+}^V & 0 \\ 0 & B_{+-}^V & D_{++}^V & 0 \\ B_{-+}^V & 0 & 0 & D_{--}^V \end{pmatrix} \qquad \tau^S = \begin{pmatrix} 0 & A_{+-}^S & C_{++}^S & 0 \\ A_{-+}^S & 0 & 0 & C_{--}^S \\ B_{++}^S & 0 & 0 & 0 \\ 0 & B_{--}^S & 0 & 0 \end{pmatrix} \tag{36}$$

When the insertion correspond to a W boson, only A_{--}^V, C_{-+}^V, B_{+-}^V, D_{++}^V are different from zero. For practical computations we have implemented a set of routines which automatically write lines of Fortran code both for the expressions of the A, B, C, D functions and for their combination to form whole spinor lines. These routines of course avoid unnecessary and time consuming multiplications by the zeroes of the τ matrices.

Before ending the section we just notice that the 4×4 matrices τ and π could also be defined exchanging indices 2 and 3, so that eqs. (35) become:

$$\tau = \begin{pmatrix} \tau_{++} & \tau_{+-} \\ \tau_{-+} & \tau_{--} \end{pmatrix} = \begin{pmatrix} A_{++} & C_{++} & A_{+-} & C_{+-} \\ B_{++} & D_{++} & B_{+-} & C_{+-} \\ A_{-+} & C_{-+} & A_{--} & C_{--} \\ B_{-+} & D_{-+} & B_{--} & D_{--} \end{pmatrix} \tag{37}$$

$$\pi_i = \begin{pmatrix} 1 & \mu_i & 0 & 0 \\ \mu_i & 1 & 0 & 0 \\ 0 & 0 & p_i^2 & \mu_i \\ 0 & 0 & \mu_i & p_i^2 \end{pmatrix} \tag{38}$$

Massless spinor lines

When one has to deal with massless spinor lines, all the formulae given in the appendix A remain valid. But in this case the fact that all μ_i's as well as m_1 and m_{n+1} are zero leads to significant simplifications. For example the rule (29) for combining pieces of spinor line is now

$$
\begin{aligned}
A^{(i+j)} &= A^{(i)} A^{(j)} + p_i^2 C^{(i)} B^{(j)} \\
B^{(i+j)} &= B^{(i)} A^{(j)} + p_i^2 D^{(i)} B^{(j)} \\
C^{(i+j)} &= A^{(i)} C^{(j)} + p_i^2 C^{(i)} D^{(j)} \\
D^{(i+j)} &= B^{(i)} C^{(j)} + p_i^2 D^{(i)} D^{(j)}
\end{aligned}
\tag{39}
$$

and the π_i matrices (34, 35) become diagonal:

$$
\pi_i =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & p_i^2 & 0 \\
0 & 0 & 0 & p_i^2
\end{pmatrix}
\tag{40}
$$

Moreover, in order to compute the whole spinor line with n insertions it is not necessary to know the whole $\tau^{(n)}$ matrix. It is clear from eq. (28) that only $A^{(n)}$ is needed. Therefore, if one for instance multiplies recursively the τ matrices of the single insertions starting from the left, at every single step one needs to compute only A and C. In fact, from eq. (39) with $i = n - 1$, $j = 1$, one sees that $B^{(n-1)}$ and $D^{(n-1)}$ are not needed to compute A^n. With $i = n - 2$, $j = 1$ one verifies that $B^{(n-2)}$ and $D^{(n-2)}$ are not needed to compute $A^{(n-1)}$ and $C^{(n-1)}$, and so on. Had one started from the right, only A and B would have had to be calculated for every product.

In most theories, like e.g. the standard model, one has to consider only vector and axial-vector couplings to massless spinor lines. This means that one has to compute only τ^V matrices (36) for every insertion. Combining together two matrices of this type, one still gets a matrix whose only elements different from zero are on the two diagonals:

$$
\begin{pmatrix}
A_{++} & 0 & 0 & C_{+-} \\
0 & A_{--} & C_{-+} & 0 \\
0 & B_{+-} & D_{++} & 0 \\
B_{-+} & 0 & 0 & D_{--}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & p_i^2 & 0 \\
0 & 0 & 0 & p_i^2
\end{pmatrix}
\begin{pmatrix}
A'_{++} & 0 & 0 & C'_{+-} \\
0 & A'_{--} & C'_{-+} & 0 \\
0 & B'_{+-} & D'_{++} & 0 \\
B'_{-+} & 0 & 0 & D'_{--}
\end{pmatrix}
=
$$

$$
=
\begin{pmatrix}
A''_{++} & 0 & 0 & C''_{+-} \\
0 & A''_{--} & C''_{-+} & 0 \\
0 & B''_{+-} & D''_{++} & 0 \\
B''_{-+} & 0 & 0 & D''_{--}
\end{pmatrix}
$$

and therefore the τ matrix of any piece of massless spinor line is 'cross-diagonal' for these theories.

Combining spinor lines. A simple example

In the following, whenever in the argument of a τ matrix or of one of its elements there will be an index μ in place of a 'polarization vector' η, we imply that the components η^ν have been taken to be $\eta^\nu = g^{\mu\nu}$. In fact every τ matrix satisfies the relation

$$\tau(p_1; \eta_1, \cdots; \eta_i, \cdots, p_n) = \eta_{i\mu}\, \tau(p_1; \eta_1, \cdots; \mu, \cdots, p_n) \tag{41}$$

and of course similar relations can be written for its A,B,C,D components. In case of just one insertion we have for example:

$$A^V_{++}(p_1; \eta, c; p_2) = \eta_\mu\, A^V_{++}(p_1; \mu, c; p_2) \tag{42}$$

From eq. (A.2) it is immediate to see that

$$A^V_{++}(p_1; \mu, c; p_2) = c_r \left(-p_1 \cdot p_2\, k_0^\mu + k_0 \cdot p_1\, p_2^\mu + k_0 \cdot p_2\, p_1^\mu - i\epsilon_{\mu\nu\rho\sigma}\, k_0^\nu\, p_1^\rho\, p_2^\sigma\right) \tag{43}$$

with $\epsilon_{0123} = 1$.

As a simple example of how to use in practice the method just exposed, let us consider the process $e^+ e^- \to t\bar{t}g$. It is described by the diagrams in fig. 2.
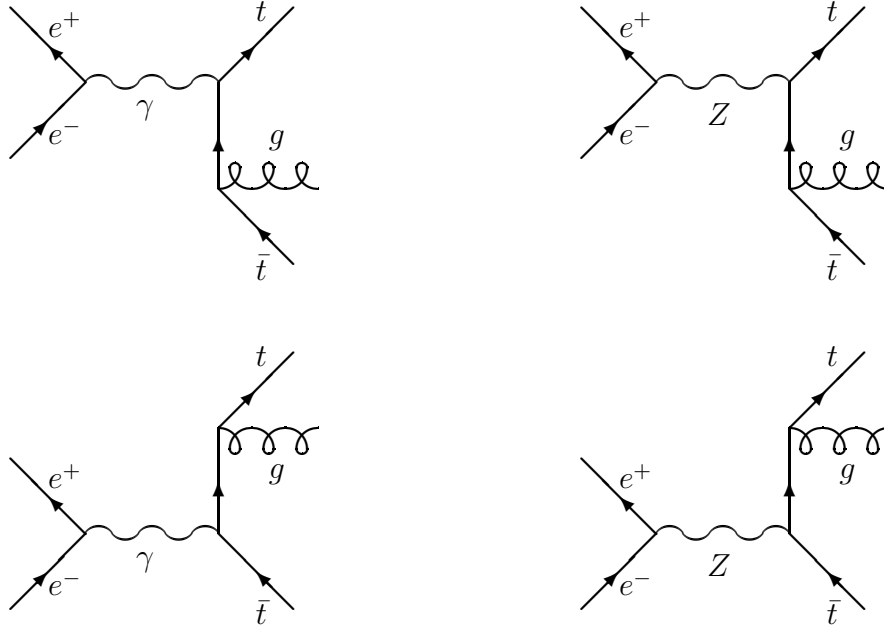

Fig. 2

In this case all insertions on spinor lines correspond to vector particles. We will indicate the four momenta of the particles with their names, so that $s = (e^+ + e^-)^2$. We will also denote with c^{fV} the couplings of a generic fermion f to a vector particle V.
One has to choose k_0 and k_1 and the form of the polarization vectors η_i^g for the gluon in terms of his momentum. One can for instance use $k_0 = (1,1,0,0)$, $k_1 = (0,0,1,0)$ and

the real polarization vectors proposed in ref. [10] or normal helicity eigenvectors. One then starts computing with eqs. (A.2) $A_{jj}(e^+; \mu, c^{e\gamma}; e^-)$ and $A_{jj}(e^+; \mu, c^{eZ}; e^-)$, with $j(= +, -)$ the polarization of the electron. With them one constructs the 'polarization' vectors

$$\eta_{j\mu}^{\gamma} = \frac{A_{jj}(e^+; \mu, c^{e\gamma}; e^-)}{s} \qquad \eta_{j\mu}^{Z} = \frac{A_{jj}(e^+; \mu, c^{eZ}; e^-)}{s - m_Z^2 + i\Gamma_Z m_Z} \qquad (44)$$

Using again eqs. (A.2), it is easy to compute the tau matrices relative to the insertions in the upper and lower part of the line:

$$\begin{aligned}
\tau_\gamma^u[j] &= \tau(t; \eta_j^\gamma, c^{t\gamma}; -\bar{t} - g) \\
\tau_Z^u[j] &= \tau(t; \eta_j^Z, c^{tZ}; -\bar{t} - g) \\
\tau_g^u[i] &= \tau(t; \eta_i^g, c^{tg}; t + g) \\
\tau_\gamma^d[j] &= \tau(t + g; \eta_j^\gamma, c^{t\gamma}; \bar{t}) \\
\tau_Z^d[j] &= \tau(t + g; \eta_j^Z, c^{tZ}; \bar{t}) \\
\tau_g^d[i] &= \tau(-\bar{t} - g; \eta_i^g, c^{tg}; \bar{t})
\end{aligned} \qquad (45)$$

and then the sum of τ_γ and τ_Z matrices:

$$\tau_{\gamma Z}^u[j] = \tau_\gamma^u[j] + \tau_Z^u[j] \qquad \tau_{\gamma Z}^d[j] = \tau_\gamma^d[j] + \tau_Z^d[j] \qquad (46)$$

The final τ matrices are obtained just composing with the law (27,31) $\tau_{\gamma Z}^u[j]$ with $\tau_g^d[i]$ and $\tau_g^u[i]$ with $\tau_{\gamma Z}^d[j]$:

$$\tau_1[i, j] = \tau_{\gamma Z}^u[j] \bullet \tau_g^d[i] \qquad \tau_2[i, j] = \tau_g^u[i] \bullet \tau_{\gamma Z}^d[j] \qquad (47)$$

From $\tau[i, j]$ the polarized amplitude is then obtained with the help of eq. (28). Indicating with l the polarization of the top and with m that of the antitop, one has

$$\begin{aligned}
\widetilde{T}_1[i, j, l, m] &= A_{1lm}[i, j] + m_t B_{1lm}[i, j] - m_t C_{1lm}[i, j] - m_t^2 D_{1lm}[i, j] \\
\widetilde{T}_2[i, j, l, m] &= A_{2lm}[i, j] + m_t B_{2lm}[i, j] - m_t C_{2lm}[i, j] - m_t^2 D_{2lm}[i, j]
\end{aligned} \qquad (48)$$

and the amplitude

$$\begin{aligned}
\text{Amp } [i, j, l, m] &= \frac{1}{\sqrt{e^+ \cdot k_0} \sqrt{e^- \cdot k_0} \sqrt{t \cdot k_0} \sqrt{\bar{t} \cdot k_0}} \\
&\times \left(\frac{\widetilde{T}_1[i, j, l, m]}{(-\bar{t} - g) \cdot k_0 \left((-\bar{t} - g)^2 - m_t^2 + i\Gamma_t m_t\right)} + \frac{\widetilde{T}_2[i, j, l, m]}{(t + g) \cdot k_0 \left((t + g)^2 - m_t^2 + i\Gamma_t m_t\right)} \right)
\end{aligned} \qquad (49)$$

If it were necessary to keep into account the electron mass, one would have to compute all the quantities $A_{ij}(e^+; \mu, c^{e\gamma}; e^-)$, $B_{ij}(e^+; \mu, c^{e\gamma}; e^-)$, $C_{ij}(e^+; \mu, c^{e\gamma}; e^-)$, $D_{ij}(e^+; \mu, c^{e\gamma}; e^-)$ and those with c^{eZ} replacing $c^{e\gamma}$. Of course i and j, the positron and electron polarization, can be different in this case. From these one then obtains

$$\begin{aligned}
\eta_{ij\mu}^{\prime\gamma} &= A_{ij}(e^+; \mu, c^{e\gamma}; e^-) - m_e B_{ij}(e^+; \mu, c^{e\gamma}; e^-) \\
&\quad + m_e C_{ij}(e^+; \mu, c^{e\gamma}; e^-) - m_e^2 D_{ij}(e^+; \mu, c^{e\gamma}; e^-)
\end{aligned} \qquad (50)$$

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$$\eta'^Z_{ij\mu} = A_{ij}(e^+; \mu, c^{eZ}; e^-) - m_e B_{ij}(e^+; \mu, c^{eZ}; e^-)$$
$$+ m_e C_{ij}(e^+; \mu, c^{eZ}; e^-) - m_e^2 D_{ij}(e^+; \mu, c^{eZ}; e^-) \qquad (51)$$

and, in the unitary gauge with $k = e^+ + e^-$,

$$\eta^\gamma_{ij\mu} = \frac{\eta'^\gamma_{ij\mu}}{s} \qquad\qquad \eta^Z_{ij\mu} = \frac{\eta'^Z_{ij\mu} - \eta'^Z_{ij} \cdot k\, k_\mu / m_Z^2}{s - m_Z^2 + i\Gamma_Z m_Z}. \qquad (52)$$

The rest of the computation is performed in analogy with the massless electron case.

In general, when one has two spinor lines connected by a vector boson, instead of computing the 'polarization' η_μ of one of the lines and use it in the other, one could compute directly the quantity $\widetilde{T}_{\lambda_1\lambda_2}(p_1; \mu, c; p_2)\ \widetilde{T}_{\lambda_3\lambda_4}(p_3; \mu, c'; p_4)$ where p_1, p_2 (p_3, p_4) are the momenta of the first (second) spinor line contiguous to the insertion. This quantity is the analog of the Z functions of ref. [7, 12]: $Z_{\lambda_1\lambda_2\lambda_3\lambda_4} = T_{\lambda_1\lambda_2}(p_1; \mu, c; p_2)\ T_{\lambda_3\lambda_4}(p_3; \mu, c'; p_4)$, and it is linear in all four (possibly unphysical) masses. In the present formalism one can immediately derive the formulae for the coefficients of the products of masses just contracting $A_{\lambda_1\lambda_2}(p_1; \mu, c; p_2)$, $B_{\lambda_1\lambda_2}(p_1; \mu, c; p_2)$, $C_{\lambda_1\lambda_2}(p_1; \mu, c; p_2)$, $D_{\lambda_1\lambda_2}(p_1; \mu, c; p_2)$, with the corresponding $A_{\lambda_3\lambda_4}(p_3; \mu, c'; p_4)$, $B_{\lambda_3\lambda_4}(p_3; \mu, c'; p_4)$, $C_{\lambda_3\lambda_4}(p_3; \mu, c'; p_4)$, $D_{\lambda_3\lambda_4}(p_3; \mu, c'; p_4)$. Using some algebra to reexpress the determinants (A.3) which appear after the contraction, the formulae coincide with the ones one can easily deduce from the expressions of the Z functions which can be found in the literature [12, 13]. This way of combining the spinor lines is not, in our opinion, the most convenient one when both spinor lines have several insertions and when the vector line connecting them has itself some insertions, due for istance to some triple vector coupling.

Comments and conclusions

The method we have exposed has been tested in several computations of physical amplitudes, both unpolarized and partially or fully polarized. Our results have always been in perfect agreement with those obtained in other formalisms.

Even if we use for external fermions the same spinors as in [7], we do not have to use also the polarizations they suggest for vector particles. In effect, expecially for massive vector particles, we often use the real polarization vectors suggested in [10]. If one defines $k_0 = (1, 0, 0, -1), k_1 = (0, 1, 0, 0)$, the massless spinors of ref. [7] coincide with those of ref. [10]. This implies that with this choice of k_0 and k_1 and the polarization for vector particles of ref. [10] our results must agree, in the limit of spinor masses going to zero, with those obtained with the method of ref. [10] for every single diagram and every polarization of external particles. This has been checked on several examples and it may be used as a valid test of the correctness of the results.

We believe that our method has some advantages with respect to those in [7, 10]. There is a certain similarity with the method of Hagiwara and Zeppenfeld since both make use of 4×4 matrices, which however take indexes in very different spaces. In our case only one τ matrix for each insertion has to be computed, instead of one for each vertex and one for each propagator. Moreover, we can freely choose the auxiliary

vectors k_0 and k_1 which can be useful both in simplifying the expressions generated in intermediate stages and as a test. Compared with the method of ref. [7, 12], our formalism is much more compact. It avoids the proliferation of terms generated by the expansion of the momenta flowing in fermion propagators in terms of the external momenta. Even with respect to the more efficient method suggested in ref. [8] for treating internal propagators, we have a smaller number of terms. The relationship between the matrix elements of u-tipe and v-type spinors (22,23,28) leads to simpler expressions than the introduction of additional auxiliary vectors, expecially when long fermion lines are present and the insertions do not all correspond to external particles. As a consequence, it is much easier in our formalism to keep track of partial results and to set up recursive schemes of evaluation which compute and store for later use subdiagrams of increasing size and complexity. Our method is also more flexible in the choice of the polarization vectors for external vector particles and one can avoid the extra integration needed to obtain the correct sum over polarization of the formalism of ref. [7, 12]. It allows to directly compute cross sections with polarized W's and Z's for any desired polarization.

The size of the matrix elements which have to be computed is steadily increasing and, despite the fast improvements in computer performance, speed is a vital issue of any method of doing calculations. We have compared our formalism with those just mentioned, avoiding in all cases to use subroutine and function statements which considerably increase computing time. Our method was consistently two to four times faster.

In conclusion, we have shown that it is possible to insert in spinor lines completeness relations in order to diagonalize the operators $\not{p}$ of the propagators of momentum p. This is particularly convenient when one uses the spinors [7]. We have presented the formalism necessary to perform actual calculations, both in the massless and in the massive case.

Appendix A

Using the spinors (4 6,14), their products can be written as

$$U(p_2, \lambda_2)\overline{U}(p_1, \lambda_1) = \frac{1}{4\sqrt{2\,p_1 \cdot k_0}\sqrt{2\,p_2 \cdot k_0}} \tag{A.1}$$

$$\times (\not{p}_2 + M_2)\left[(1 + \lambda_1\lambda_2) - (\lambda_1 + \lambda_2)\gamma^5 + \not{k}_1[(\lambda_1 - \lambda_2) - (1 - \lambda_1\lambda_2)\gamma^5]\right]\not{k}_0(\not{p}_1 + M_1).$$

Multiplying to the right by χ^S or χ^V of eqs. (16,17) , taking the trace and with the help of eq. (22), one gets the following expressions for the functions A, B, C, D corresponding to a scalar and a vector insertion :

$$
\begin{aligned}
A^S_{+-} &= c_l\,(k_0 \cdot p_1 \; k_1 \cdot p_2 - k_0 \cdot p_2 \; k_1 \cdot p_1 - i\epsilon\,(k_0, k_1, p_1, p_2)) \\
A^S_{-+} &= c_r\,(-k_0 \cdot p_1 \; k_1 \cdot p_2 + k_0 \cdot p_2 \; k_1 \cdot p_1 - i\epsilon\,(k_0, k_1, p_1, p_2)) \\
B^S_{++} &= c_r\,k_0 \cdot p_2 \\
B^S_{--} &= c_l\,k_0 \cdot p_2 \\
C^S_{++} &= c_l\,k_0 \cdot p_1 \\
C^S_{--} &= c_r\,k_0 \cdot p_1 \\
A^V_{++} &= c_r\,(-k_0 \cdot \eta \; p_1 \cdot p_2 + k_0 \cdot p_1 \; \eta \cdot p_2 + k_0 \cdot p_2 \; \eta \cdot p_1 + i\epsilon\,(k_0, \eta, p_1, p_2)) \qquad \text{(A.2)}\\
A^V_{--} &= c_l\,(-k_0 \cdot \eta \; p_1 \cdot p_2 + k_0 \cdot p_1 \; \eta \cdot p_2 + k_0 \cdot p_2 \; \eta \cdot p_1 - i\epsilon\,(k_0, \eta, p_1, p_2)) \\
B^V_{+-} &= c_l\,(k_0 \cdot \eta \; k_1 \cdot p_2 - k_0 \cdot p_2 \; k_1 \cdot \eta - i\epsilon\,(k_0, k_1, \eta, p_2)) \\
B^V_{-+} &= c_r\,(-k_0 \cdot \eta \; k_1 \cdot p_2 + k_0 \cdot p_2 \; k_1 \cdot \eta - i\epsilon\,(k_0, k_1, \eta, p_2)) \\
C^V_{+-} &= c_r\,(-k_0 \cdot \eta \; k_1 \cdot p_1 + k_0 \cdot p_1 \; k_1 \cdot \eta + i\epsilon\,(k_0, k_1, \eta, p_1)) \\
C^V_{-+} &= c_l\,(k_0 \cdot \eta \; k_1 \cdot p_1 - k_0 \cdot p_1 \; k_1 \cdot \eta + i\epsilon\,(k_0, k_1, \eta, p_1)) \\
D^V_{++} &= c_l\,k_0 \cdot \eta \\
D^V_{--} &= c_r\,k_0 \cdot \eta.
\end{aligned}
$$

All functions A, B, C, D for a single insertion not reported in the preceding list are identically zero.
The function ϵ is defined to be the determinant:

$$\epsilon\,(p, q, r, s) = det\begin{vmatrix} p^0 & q^0 & r^0 & s^0 \\ p^1 & q^1 & r^1 & s^1 \\ p^2 & q^2 & r^2 & s^2 \\ p^3 & q^3 & r^3 & s^3 \end{vmatrix} \tag{A.3}$$

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